

Cone Volumes

Teacher Sheet

Introduction

The aim of this activity is to have students work out the volume of a cone using a variety of different known measurements. This involves application of Pythagoras' Theorem, and right-angled triangle Trigonometry.

Then the students progress to finding the optimum apex angle to maximise the volume of a cone for a given slant height. **No knowledge of calculus** is required – the process is completed using numeric Graph Analysis tools.

This activity is designed for students aged 12 to 15, both as a consolidating activity of existing skills and preparing the conceptual way for future optimisation problems.

Although the screenshots in this document are taken from a colour screen Nspire CX, the activity works just as well on a greyscale Nspire handheld. However, OS 3.0.2 or later is required.

Problem 1 – Introductions and Assumptions

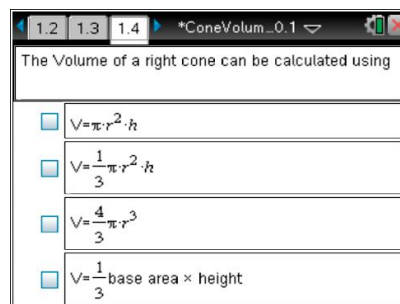
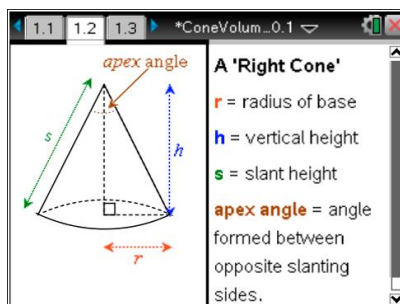
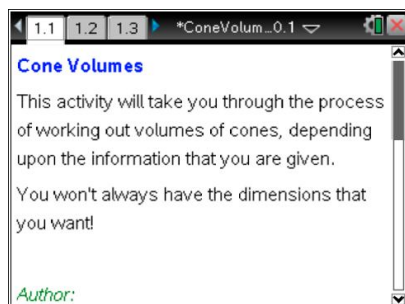
The first 11 pages introduce the vocabulary that will be used, and checks the students' existing knowledge and ability to apply Pythagoras (pages 1.6 and 1.7) and Trigonometry (pages 1.8 to 1.11). In order to answer the questions on these pages, students will most likely need to draw their own diagrams on their page, as well as use the Scratchpad for the evaluation of their calculations. *The Scratchpad Calculator can be accessed by pressing ⌘ , and afterwards press esc to return to the document.*

In the process of working out the numerical answers, students may report that they obtain a similar, but not exactly the same decimal answers as those offered as options. This situation can arise from using, say, 3.14 instead of the exact value of π , or by not using the full decimal value of a previous answer. If this occurs, students should be **shown by the teacher** how to use the Scratchpad to generate answers that not only use π , but that also do not suffer from rounding errors.

Students also need to be aware of whether their Scratchpad settings are in **Degrees or Radians**. It can be left to the teacher to decide whether to forewarn the students of this, or to let them have the learning experience of realising this themselves.

The question pages have been designated to be 'Self-Check', so students should press ⌘ then \blacktriangle to verify if they are correct, after each one.

Students should note that page 1.4 has square option boxes, meaning that more than one option is correct, and that they should select **all** the correct formulae in order for it to be marked correct.



1.3 1.4 1.5 *ConeVolum...0.1

A right cone has base diameter 8cm and height 10cm. Its volume is...

- 167.6 cm³ (1 dp)
- 251.3 cm³ (1 dp)
- 502.7 cm³ (1 dp)
- 670.2 cm³ (1 dp)

1.6 1.7 1.8 *ConeVolum...0.1

A right cone has radius 5cm and vertical height 8cm. What is the angle at its apex?

- 32.0° (1 dp)
- 58.0° (1 dp)
- 64.0° (1 dp)
- 77.4° (1 dp)
- 102.6° (1 dp)
- 116.0° (1 dp)

1.4 1.5 1.6 *ConeVolum...0.1

A right cone has circular base radius 6.5cm and slant height of 11cm. Its volume is...

- 392.6 cm³ (1 dp)
- 486.7 cm³ (1 dp)
- 565.3 cm³ (1 dp)
- 3484.2 cm³ (1 dp)

1.7 1.8 1.9 *ConeVolum...0.1

A right cone has slant height 15cm and vertical height 12cm. What is the angle at its apex?

- 73.7° (1 dp)
- 77.3° (1 dp)
- 102.7° (1 dp)
- 106.3° (1 dp)

1.5 1.6 1.7 *ConeVolum...0.1

A circular based right cone has height 25cm and slant height 30cm. Its volume is...

- 434.1 cm³ (1 dp)
- 1022.4 cm³ (1 dp)
- 7199.5 cm³ (1 dp)
- 8639.4 cm³ (1 dp)
- 39924.4 cm³ (1 dp)

1.8 1.9 1.10 *ConeVolum...0.1

The apex angle of a right cone is 48° and its slant height is 11cm. How tall is it?

- 4.5 cm (1 dp)
- 7.4 cm (1 dp)
- 8.2 cm (1 dp)
- 10.0 cm (1 dp)

1.9 1.10 1.11 *ConeVolum...0.1

The apex angle of a right cone is 85° and its slant height is 17cm. Its volume is...

- 150.7 cm³ (1 dp)
- 445.0 cm³ (1 dp)
- 1731.3 cm³ (1 dp)
- 2348.2 cm³ (1 dp)

The incorrect, distractor answers on each page are all generated from making 'Classic Mistakes' such as adding instead of subtracting when using Pythagoras' Theorem, using the incorrect trigonometric ratio, not halving the apex angle when dealing with the angle inside the right-angled triangle, etc.

Problems 2, 3 and 4 – Focusing on the Apex Angle in Context

Pages 2.1 and 2.2 introduce problems 3 and 4. On pages 3.1 and 4.2, students should grab the white circles to align the segments with the sides of the cone, and thus read off the accurate apex angle.

1.10 1.11 2.1 *ConeVolum...0.1

It is not always easy to measure the radius and vertical height of a cone.

It's often easier to measure the apex angle and slant height. **Especially** if you only have a photograph of a cone to work from....

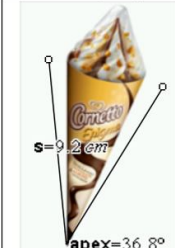
Turn to the next page...

1.11 2.1 2.2 *ConeVolum...0.1

On the following pages, grab and move the angle markers to fit the sides of each cone.

Then, using the apex angle and slant height shown, write down the required workings on your page to verify that the volume answer is correct.

2.1 2.2 3.1 *ConeVolum...0.1



Grab and move the white circles to match the cone edges.

Apex angle=36.8°

Slant height, s=9.2 cm

Volume=77.1 cm³

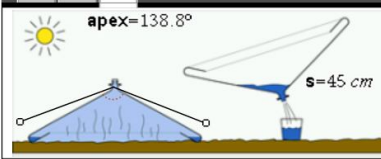
2.2 3.1 4.1 *ConeVolum...0.1

The Water Cone™ on the next page is used in developing worlds to obtain clean drinking water from salt water, by the process of evaporation.

See www.thewatercone.com for more info.

As you did for the Cornetto, verify that the volume of the Water Cone™ is correct.

3.1 4.1 4.2 *ConeVolum...0.1



Apex angle=138.8° Slant height, s=45 cm

Volume=29418.5 cm³

Problem 5 – Optimising the Cone Volume

Page 5.2 is important for students to experiment with, to appreciate that the maximum volume comes when the apex angle is around 110° . When the apex angle is set to 0° or 180° , are the students surprised by what they see (or don't see!) ?

Page 5.3 could be used for class discussion about why the Cornetto and WaterCone™ were not designed to maximise the volume of their cones.

4.1 4.2 5.1 *ConeVolum...0.1

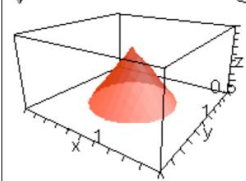
For a given slant height, you can create cones with different apex angles.

The next page allows you to view many such cones, each with slant height 5cm.

Vary the apex angle using the **slider** in the top left corner. To rotate the view of your cone, first press **esc** to 'exit' the slider, then use the arrow keys.

4.2 5.1 5.2 *ConeVolum...0.1

apex = 75.



Apex angle = $75.^\circ$

Volume = 38.5 cm^3

Rotate the view using arrow keys.

5.1 5.2 5.3 *ConeVolum...0.1

Go back to the previous page 5.2 and adjust the apex angle so that the volume is as large as possible. **Then consider....**

- Has the Cornetto been designed to hold the maximum volume of ice cream?
- Has the WaterCone™ been designed to hold the maximum volume of air?

Page 5.4 introduces the task of optimising the volume, whilst pages 5.5 to 5.8 take students step-by-step towards deriving the algebraic function for the volume in terms of slant height, s , and apex angle, x .

Page 5.8 requires students to correctly identify **all** of the valid algebraic versions (more than one is correct)

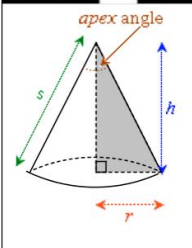
5.2 5.3 5.4 *ConeVolum...0.1

You will now aim to accurately find the optimum apex angle that maximises the volume, for a given slant height.

You have already done all the required thinking of the process when you verified the volumes of the Cornetto and the WaterCone.

You also know roughly what the best apex angle should be, from your 3D cone analysis.

5.3 5.4 5.5 *ConeVolum...0.1



The key to the whole process is knowing the angles and lengths of the sides of the shaded triangle.

let apex angle = x

5.4 5.5 5.6 *ConeVolum...0.1

If the slant height is 's' and the apex angle is 'x', then the radius, r, of the cone is

- ☐ $r = s \cdot \cos\left(\frac{x}{2}\right)$
- ☐ $r = s \cdot \cos(x)$
- ☐ $r = s \cdot \sin\left(\frac{x}{2}\right)$
- ☐ $r = s \cdot \sin(x)$

5.5 5.6 5.7 *ConeVolum...0.1

If the slant height is 's' and the apex angle is 'x', then the height, h, of the cone is

- ☐ $h = s \cdot \cos\left(\frac{x}{2}\right)$
- ☐ $h = s \cdot \cos(x)$
- ☐ $h = s \cdot \sin\left(\frac{x}{2}\right)$
- ☐ $h = s \cdot \sin(x)$

5.6 5.7 5.8 *ConeVolum...0.1

Therefore the Volume of the cone can be written as:

- ☐ $V = \frac{1}{3} \pi \left(s \cdot \sin\left(\frac{x}{2}\right) \right)^2 \left(s \cdot \cos\left(\frac{x}{2}\right) \right)$
- ☐ $V = \frac{\pi}{3} s^2 \cdot \left(\sin\left(\frac{x}{2}\right) \right)^2 \cdot \cos\left(\frac{x}{2}\right)$
- ☐ $V = \frac{\pi}{3} s^3 \cdot \left(\sin\left(\frac{x}{2}\right) \right)^2 \cdot \cos\left(\frac{x}{2}\right)$

5.7 5.8 5.9 *ConeVolum...0.1

You will now use the formula that you have derived and checked on the previous page.

Let the slant height, **s = 5cm**.

This will allow you to compare your results to the 3D cone volume answers, on page 5.5

Scroll down....↓

On the next page, graph the formula for the volume of the cone, based on knowing the apex angle, x .

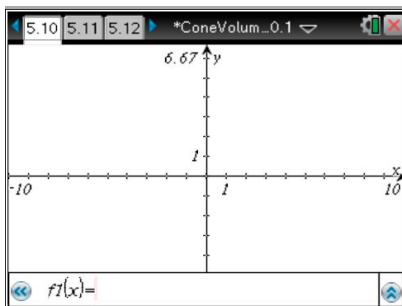
You **will** need to adjust the axes to view the section of the graph that shows you the optimum apex angle, x .

You **may** also need to change the Graphs & Geometry settings for the Graphing Angle to **Degrees**.

You will also need to use the **menu > Analyse Graph** tools to locate the optimum apex angle. **Find it to 2 decimal places.**

When analysing the graph on page 5.10, students need to know how to use the 'maximum' option.

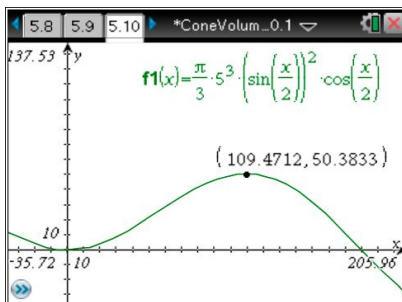
To display the answer to 2 decimal places, one way is to move over the x co-ordinate of the maximum turning point and then press the **⊕** key a few times.



Decide which of the following statements is true by altering the value of s , on the previous page.

- ☐ If you change the length of the slant height, then the optimum apex angle also changes.
- ☐ The optimum apex angle is NOT affected by the length of the slant height.

You should find that the apex angle that gives the maximum volume of a cone is approximately 109.5° .
This seems a strange number as a result.
However, this is not strange to the world of Chemistry!
 Use the Internet to find out what you can about "109.5 degrees"



The completed graph is shown on the left, for when the slant height, $s = 5$ cm.
 The maximum volume of $50.8333... \text{ cm}^3$ comes when the apex angle is $109.4712....$ degrees.

Problem 6 – Light Hearted Ending, with Extension Task.

Pages 6.1 to 6.4 are meant to be a light-hearted ending to the task.

Page 6.5 can be used as an extension task for able students who finish early to attempt to solve. Again, more than one correct answer exists on this page. Once the correct answers have been identified, students could then embark upon deriving these formulae from scratch – this might take another significant chunk of time.

The inventor of the orange traffic cone was...

- ☐ an American
- ☐ an Australian
- ☐ a Brit
- ☐ a Frenchman

The traffic cone was invented...

- ☐ at the start of World War 1
- ☐ at the start of World War 2
- ☐ at the end of World War 2
- ☐ during the 1960's

Ice cream cones were invented...

- ☐ in America, in 1904
- ☐ in Italy, in 1907
- ☐ in Spain, in 1913

How many years after they were first invented, were ice-cream cones finally mass produced?

- ☐ 5 years
- ☐ 10 years
- ☐ 20 years
- ☐ 30 years

EXTENSION QUESTION:
 The curved surface area of a cone is given by:

- ☐ $A = \pi r s$
- ☐ $A = \frac{1}{2} \pi r h$
- ☐ $A = \pi r \sqrt{r^2 + h^2}$
- ☐ $A = \pi r^2 \div \tan\left(\frac{\text{apex angle}}{2}\right)$

Acknowledgements

Images used in this activity were sourced from the following internet pages on 18 June 2011.

Page 3.1

http://www.fdin.org.uk/wordpress/wp-content/uploads/Cornetto_Enigma_VanChoc_Wrap.jpg

Page 4.2

<http://greentechfreedom.com/wp-content/uploads/2010/11/funktion1600.jpg>